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Algebraic Bethe ansatz for the anisotropic supersymmetric U model

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Abstract. We present an algebraic Bethe ansatz for the anisotropic supersymmetric U model for correlated electrons on the unrestricted 4^L-dimensional electronic Hilbert space $\otimes_{n=1}^{L} C^4$ (where L is the lattice length). The supersymmetry algebra of the local Hamiltonian is the quantum superalgebra $U_q[gl(2|1)]$ and the model contains two symmetry-preserving free real parameters; the quantization parameter q and the Hubbard interaction parameter U. The parameter U arises from the one-parameter family of inequivalent typical four-dimensional irreps of $U_q[gl(2|1)]$. Eigenstates of the model are determined by the algebraic Bethe ansatz on a one-dimensional periodic lattice.

1. Introduction

The supersymmetic U model [4] is an example of a generalized Hubbard model which is integrable in one-dimension. Integrability of the model was shown through the quantum inverse scattering method (QISM) using an R-matrix associated with the one parameter family of typical four-dimensional representations of the Lie superalgebra $gl(2|1)$. As a consequence the model is $gl(2|1)$ invariant and continuously depends on one free parameter, compatible with the integrability, arising from the underlying representation. Bethe ansatz solutions of this model have been studied in [11, 9, 12].

In [13] an anisotropic generalization of the supersymmetic U model was proposed and solved by means of the co-ordinate Bethe ansatz. In this model an additional anisotropy parameter for correlated hopping terms was introduced which then produces a model with two free parameters. It was subsequently shown [1, 10] that the anisotropic model can be derived in the framework of the QISM, thus demonstrating that the model is in fact *integrable*; that is, there exists an infinite number of conservation laws for the system. In this case the model was derived from a $U_q[gl(2|1)]$ invariant R-matrix, relating the anisotropy parameter with the deformation parameter q of the sypersymmetry algebra.

Not only does the QISM provide a means to derive integrable models, it also provides a framework for an *algebraic* Bethe ansatz to be applied for the solution of such models. Our aim in this paper is to pursue such a formulation for the anisotropic supersymmetric U model. This approach has been discussed in [2] for a system derived from an abstract $U_q[osp(2|2)] \cong U_q[gl(2|1)]$ invariant R-matrix. However, nearly all technical details were not presented. Here we wish to provide a rigorous derivation of the Bethe ansatz equations (BAE), confirming those given in [13, 2].

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We now introduce some notation as in [4]. Electrons on a lattice are described by canonical Fermi operators $c_{i,\sigma}$ and $c_{i,\sigma}^\dagger$ satisfying the anti-commutation relations given by $\{c_{i,\sigma}^\dagger, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}$, where $i, j = 1, 2, .., L$ and $\sigma, \tau = \uparrow, \downarrow$. The operator $c_{i,\sigma}$ annihilates an electron of spin σ at site i, which implies that the Fock vacuum $|0\rangle$ satisfies $c_{i,\sigma}|0\rangle = 0$. At a given lattice site i there are four possible electronic states:

$$|0\rangle \qquad |\uparrow\rangle_i = c_{i,\uparrow}^\dagger|0\rangle \qquad |\downarrow\rangle_i = c_{i,\downarrow}^\dagger|0\rangle \qquad |\uparrow\downarrow\rangle_i = c_{i,\downarrow}^\dagger c_{i,\uparrow}^\dagger|0\rangle.$$

By $n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}$ we denote the number operator for electrons with spin σ on site i, and we write $n_i = n_{i,\uparrow} + n_{i,\downarrow}$. The global Hamiltonian for this model on a one-dimensional periodic lattice as given in [13] is

$$H = -\sum_{i,\sigma}(c_{i\sigma}^\dagger c_{i+1\sigma} + h.c.)\exp\left[-\frac{1}{2}(\zeta - \sigma\gamma)n_{i,-\sigma} - \frac{1}{2}(\zeta + \sigma\gamma)n_{i+1,-\sigma}\right]$$
$$+\sum_i\left[Un_{i\uparrow}n_{i\downarrow} + t(c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger c_{i+1\downarrow}c_{i+1\uparrow} + h.c.)\right] \qquad (1)$$

where i labels the sites and

$$t = \frac{U}{2} = \epsilon[2e^{-\zeta}\cosh\zeta - \cosh\gamma)]^{\frac{1}{2}} \qquad \epsilon = \pm 1.$$

The Hamiltonian may be obtained from the R-matrix for the one-parameter family of four-dimensional representations of $U_q[gl(2|1)]$, which is afforded by the module W with highest weight $(0, 0|\alpha)$. For $\alpha > 0$ or $\alpha < -1$, the module W is unitary and thus the tensor product $W \otimes W$ is completely reducible [3]. We write $W \otimes W = W_1 \oplus W_2 \oplus W_3$, where W_1, W_2 and W_3 are $U_q[gl(2|1)]$-modules with highest weights $(0, 0|2\alpha)$, $(0, -1|2\alpha+1)$ and $(-1, -1|2\alpha+2)$, respectively. Let P_k, $k = 1, 2, 3$ be the projection operator from $W \otimes W$ onto W_k. The trigonometric R-matrix, which satisfies the quantum Yang–Baxter equation, was given in [4] in the form

$$\check{R}(x) = \frac{q^x - q^{2x}}{1 - q^{x+2\alpha}}P_1 + P_2 + \frac{1 - q^{x+2\alpha+2}}{q^x - q^{2\alpha+2}}P_3.$$

The local Hamiltonian is given by [5]

$$H_{i,i+1} = -\frac{(q^{\alpha+1} - q^{-\alpha-1})}{\ln q}\frac{d}{dx}\check{R}_{i,i+1}(x)\big|_{x=0}$$

with

$$e^\gamma = q \qquad e^\zeta = \frac{[\alpha+1]_q}{[\alpha]_q} \qquad U = 2[\alpha]_q^{-1}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad x \in C.$$

We assume throughout that $q \in \mathbb{R}$ and $q > 0$. We stress that the Hamiltonian is Hermitian only for $U > -2$; that is, when the underlying $U_q[gl(2|1)]$ representation is unitary.

The global Hamiltonian H is solvable by means of the QISM which we will demonstrate below. The quasi-triangular structure of the affine superalgebra $U_q[gl(2|1)^{(1)}]$ allows us to replace the auxilliary space W with the vector representation space V to simplify the calculation of the *nested algebraic Bethe ansatz* (NABA).

The paper is set out as follows. The graded QISM will be discussed in section 2. The use of the QISM enables us to obtain expressions for an infinite number of higher conservation laws at the quantum level. Section 4 will be the construction of the algebraic Bethe ansatz for the model. We formulate a set of simultaneous eigenstates of the transfer matrix using a NABA. The expression obtained for the BAE will be compared with those given in [2].

2. Graded quantum inverse scattering method

We will construct eigenstates of the Hamiltonian of the one-dimensional supersymmetric model above, using the QISM. The supersymmetry of the model requires a modification of the QISM. We use the R-matrix satisfying the graded Yang–Baxter equation and introduce an L operator constructed directly from the R-matrix of the twisted representation.

The graded Yang–Baxter equation can be written as the operator equation:

$$\begin{aligned} R_{\alpha_1\beta_1,\alpha_2\beta_2}(x/y)L(x)_{\beta_1\gamma_1ab}L(y)_{\beta_2\gamma_2bc}(-1)^{[\beta_2]([\beta_1]+[\gamma_1])} \\ = L(y)_{\alpha_2\beta_2ab}L(x)_{\alpha_1\beta_1bc}(-1)^{[\beta_2]([\alpha_1]+[\beta_1])}R_{\beta_1\gamma_1,\beta_2\gamma_2}(x/y) \end{aligned} \tag{2}$$

acting on the spaces $V\otimes V\otimes W$ where V is the vector module and W is a four-dimensional module of inequivalent irreps. Greek indices are used to label the matrix spaces, that is the first two spaces and Roman indices label the quantum space, which is the third space. The quantum space represents the Hilbert space over a site on the one-dimensional lattice. The R-matrix acts in the matrix space and it is between the two matrix spaces that the graded tensor product acts.

The R-matrix acts on $V\otimes V$ and has the form [14, 7]

$$R(x)=\begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 & -C & 0 & 0 \\ 0 & xC & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & -C & 0 \\ 0 & 0 & xC & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xC & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

where $A(x)=\frac{1-xq^2}{x-q^2}$, $E(x)=\frac{(1-x)q}{x-q^2}$ and $C(x)=\frac{1-q^2}{x-q^2}$ which satisfies the Yang–Baxter equation. The L operator is constructed in the next section.

3. The L operator

The L operator will be constructed from a $V\otimes W$ representation where as before V denotes the vector module and W corresponds to the one parameter family of the inequivalent typical four-dimensional irreps. The weights for module V are $(1,0|0)$, $(0,1|0)$, $(0,0|1)$ with the corresponding weight basis $|1\rangle$, $|2\rangle$, and $|3\rangle$ respectively. On this module the $U_q[gl(2|1)]$ generators act as $E^i_j=e^i_j$. We choose the grading for module V to be

$$[1]=[2]=0 \qquad [3]=1.$$

The weights for module W are $(0,0|\alpha)$, $(0,-1|\alpha+1)$,$(-1,0|\alpha+1)$ and $(-1,-1|\alpha+2)$ respectively with basis vectors $|a\rangle$, $|b\rangle$, $|c\rangle$, $|d\rangle$. The $U_q[gl(2|1)]$ generators act as

$$\begin{aligned} E^1_1 &= -e_{cc}-e_{dd} \\ E^2_2 &= -e_{bb}-e_{dd} \end{aligned}$$

$$
\begin{aligned}
E_3^3 &= \alpha e_{aa} + (\alpha+1)(e_{bb}+e_{cc}) + (\alpha+2)e_{dd}\\
E_2^1 &= e_{bc}\\
E_1^2 &= e_{cb}\\
E_3^2 &= \sqrt{[\alpha]_q}e_{ab} + \sqrt{[\alpha+1]_q}e_{cd}\\
E_2^3 &= \sqrt{[\alpha]_q}e_{ba} + \sqrt{[\alpha+1]_q}e_{dc}.
\end{aligned}
$$

We choose the grading for the module W to be

$$[a] = [d] = 0 \qquad [b] = [c] = 1.$$

We note for values $\alpha > 0$ we have

$$(E_j^i)^\dagger = E_i^j$$

and the representation is referred to as unitary of type I. For $\alpha < -1$ we have

$$(E_j^i)^\dagger = (-1)^{[i]+[j]}E_i^j$$

and the representation is unitary of type II [3].

The tensor product decomposition is $V \otimes W = V_1 \oplus V_2$, where V_1 has highest weight $(1, 0|\alpha)$ and V_2 has highest weight $(0, 0|\alpha+1)$. Applying the Baxterization procedure described in [8] gives the $\check{R}$-matrix for this $V \otimes W$ representation as

$$\check{R}(x) = \frac{1 - xq^{-2-\alpha}}{x - q^{-2-\alpha}}\check{P}_1 + \check{P}_2.$$

In the above the $\check{P}_i$ are $U_q[gl(2|1)]$-invariant operators $\check{P}_i : V \otimes W \to W \otimes V$. We define the L operator as

$$L(x) = P\check{R}(x) = \frac{1 - xq^{-2-\alpha}}{x - q^{-2-\alpha}}P_1 - P_2 \tag{3}$$

where P_1 and P_2 are twisted intertwiners defined by $P_1 = P\check{P}_1$ and $P_2 = -P\check{P}_2$; we construct them in the following way.

The co-product $\Delta : U_q[gl(2|1)] \to U_q[gl(2|1)]$ is defined by

$$
\begin{aligned}
\Delta(E_i^i) &= 1 \otimes E_i^i + E_i^i \otimes 1 \qquad i = 1, 2, 3,\\
\Delta(E_2^1) &= E_2^1 \otimes q^{\frac{1}{2}(E_1^1-E_2^2)} + q^{-\frac{1}{2}(E_1^1-E_2^2)} \otimes E_2^1\\
\Delta(E_1^2) &= E_1^2 \otimes q^{\frac{1}{2}(E_1^1-E_2^2)} + q^{-\frac{1}{2}(E_1^1-E_2^2)} \otimes E_1^2\\
\Delta(E_3^2) &= E_3^2 \otimes q^{\frac{1}{2}(E_2^2+E_3^3)} + q^{-\frac{1}{2}(E_2^2+E_3^3)} \otimes E_3^2\\
\Delta(E_2^3) &= E_2^3 \otimes q^{\frac{1}{2}(E_2^2+E_3^3)} + q^{-\frac{1}{2}(E_2^2+E_3^3)} \otimes E_2^3.
\end{aligned}
\tag{4}
$$

Symmetry adapted orthonormal bases may be expressed in terms of $|\phi_\gamma^1\rangle$ and $|\phi_\beta^1\rangle$ for V_1 and V_2 respectively with $\gamma = 1, ..., 8$, $\beta = 1, ..., 4$.

The basis for V_1 is given by

$$
\begin{aligned}
|\Phi_8^1(q)\rangle &= [\alpha+2]_q^{-\frac{1}{2}}(q^{-\frac{1}{2}(\alpha+1)}|3\rangle \otimes |c\rangle + q^{\frac{1}{2}}\sqrt{[\alpha+1]_q}|2\rangle \otimes |d\rangle)\\
|\Phi_7^1(q)\rangle &= [\alpha+2]_q^{-\frac{1}{2}}(q^{-\frac{1}{2}(\alpha+1)}|3\rangle \otimes |b\rangle + q^{\frac{1}{2}}\sqrt{[\alpha+1]_q}|1\rangle \otimes |d\rangle)\\
|\Phi_6^1(q)\rangle &= |2\rangle \otimes |c\rangle\\
|\Phi_5^1(q)\rangle &= (q+q^{-1})^{-\frac{1}{2}}(q^{-\frac{1}{2}}|2\rangle \otimes |b\rangle + q^{\frac{1}{2}}|1\rangle \otimes |c\rangle)\\
|\Phi_4^1(q)\rangle &= [\alpha+1]_q^{-\frac{1}{2}}(q^{-\frac{\alpha}{2}}|3\rangle \otimes |a\rangle + q^{\frac{1}{2}}\sqrt{[\alpha]_q}|2\rangle \otimes |b\rangle)
\end{aligned}
$$

$$|\Phi_3^1(q)\rangle = |1\rangle \otimes |b\rangle$$
$$|\Phi_2^1(q)\rangle = |2\rangle \otimes |a\rangle$$
$$|\Phi_1^1(q)\rangle = |1\rangle \otimes |a\rangle \tag{5}$$

and the basis for V_2 is given by

$$\begin{aligned}
|\Phi_4^2(q)\rangle &= |3\rangle \otimes |d\rangle \\
|\Phi_3^2(q)\rangle &= [\alpha+2]_q^{-\frac{1}{2}}(q^{-\frac{1}{2}(\alpha+1)}|2\rangle \otimes |d\rangle - q^{\frac{1}{2}}\sqrt{[\alpha+1]_q}|3\rangle \otimes |c\rangle) \\
|\Phi_2^2(q)\rangle &= [\alpha+2]_q^{-\frac{1}{2}}(q^{-\frac{1}{2}(\alpha+1)}|1\rangle \otimes |d\rangle - q^{\frac{1}{2}}\sqrt{[\alpha+1]_q}|3\rangle \otimes |b\rangle) \\
|\Phi_1^2(q)\rangle &= [\alpha+2]_q^{-\frac{1}{2}}(q^{-\frac{1}{2}(\alpha+1)}|1\rangle \otimes |c\rangle - q^{\frac{1}{2}(1-\alpha)}|2\rangle \otimes |b\rangle \\
&\quad + q\sqrt{[\alpha]_q}|3\rangle \otimes |a\rangle).
\end{aligned} \tag{6}$$

So we may express

$$P_1 = \sum_\gamma |\Phi_\gamma^1(q^{-1})\rangle\langle\Phi_\gamma^1(q)|$$

where $\gamma = 1, ..., 8$ and

$$P_2 = \sum_\beta |\Phi_\beta^2(q^{-1})\rangle\langle\Phi_\beta^2(q)|$$

where $\beta = 1, ..., 4$ and also note the rules

$$(|x\rangle \otimes |y\rangle)^\dagger = (-1)^{[|x\rangle][|y\rangle]}\langle x| \otimes \langle y|$$
$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]}(ac \otimes bd).$$

We find P_1 to be given by

$$\begin{aligned}
P_1 = e_{11} &\otimes \left(e_{aa} + e_{bb} + e_{cc}(D+F) + \frac{[\alpha+1]_q}{[\alpha+2]_q}e_{dd}\right) \\
&+ e_{22} \otimes \left(e_{aa} + e_{cc} + e_{bb}(D+F) + \frac{[\alpha+1]_q}{[\alpha+2]_q}e_{dd}\right) \\
&+ e_{33} \otimes \left(\frac{1}{[\alpha+2]_q}(e_{bb}+e_{cc}) + G^2 e_{aa}\right) \\
&+ e_{12} \otimes e_{cb}(-qD + q^{-1}F) + e_{21} \otimes e_{bc}(-q^{-1}D + qF) \\
&+ e_{13} \otimes \left(-\frac{\sqrt{[\alpha+1]_q}q^{-\frac{\alpha}{2}-1}}{[\alpha+2]_q}e_{db} + GDq^{\frac{-\alpha-1}{2}}e_{ca}\right) \\
&+ e_{31} \otimes \left(\frac{\sqrt{[\alpha+1]_q}q^{\frac{\alpha}{2}+1}}{[\alpha+2]_q}e_{bd} - GDq^{\frac{\alpha+1}{2}}e_{ac}\right) \\
&+ e_{23} \otimes \left(-\frac{\sqrt{[\alpha+1]_q}q^{-\frac{\alpha}{2}-1}}{[\alpha+2]_q}e_{dc} - GDq^{\frac{-\alpha-3}{2}}e_{ba}\right) \\
&+ e_{32} \otimes \left(\frac{\sqrt{[\alpha+1]_q}q^{\frac{\alpha}{2}+1}}{[\alpha+2]_q}e_{cd} + GDq^{\frac{\alpha+3}{2}}e_{ab}\right)
\end{aligned}$$

where

$$D = (1 + q^{-2} + (q^{-1}+q)^2 q^{\alpha+1}[\alpha]_q^{-1})^{-\frac{1}{2}}(1 + q^2 + (q^{-1}+q)^2 q^{-\alpha-1}[\alpha]_q^{-1})^{-\frac{1}{2}}$$
$$F = (1+q^2)^{-\frac{1}{2}}(1+q^{-2})^{-\frac{1}{2}}$$

$$G = \frac{(q+q^{-1})}{\sqrt{[\alpha]_q}}$$

and P_2 to be given by

$$\begin{aligned}
[\alpha+2]_q P_2 = {} & e_{11}\otimes(e_{cc}+e_{dd}) + e_{22}\otimes(e_{bb}+e_{dd}) \\
& + e_{33}\otimes([\alpha]_q e_{aa} + [\alpha+1]_q(e_{bb}+e_{cc}) + [\alpha+2]_q e_{dd}) \\
& - q e_{12}\otimes e_{cb} - q^{-1} e_{21}\otimes e_{bc} \\
& + e_{13}\otimes(q^{\frac{\alpha}{2}+1}\sqrt{[\alpha+1]_q}e_{db} - q^{\frac{\alpha+3}{2}}\sqrt{[\alpha]_q}e_{ca}) \\
& + e_{31}\otimes(-q^{-\frac{\alpha}{2}-1}\sqrt{[\alpha+1]_q}e_{bd} + q^{\frac{-\alpha-3}{2}}\sqrt{[\alpha]_q}e_{ac}) \\
& + e_{23}\otimes(q^{\frac{\alpha}{2}+1}\sqrt{[\alpha+1]_q}e_{dc} + q^{\frac{\alpha+1}{2}}\sqrt{[\alpha]_q}e_{ba}) \\
& - e_{32}\otimes(q^{-\frac{\alpha}{2}-1}\sqrt{[\alpha+1]_q}e_{cd} + q^{\frac{-\alpha-1}{2}}\sqrt{[\alpha]_q}e_{ab}).
\end{aligned}$$

We write

$$T_L(x) = L_L(x)L_{L-1}(x)\ldots L_1(x)$$

$$[T_L(x)^{ab}]_{\alpha_1,\beta_1,\ldots,\alpha_L,\beta_L} = L_L(x)^{\alpha_L\beta_L}_{ac_L}\ldots L_1(x)^{\alpha_1\beta_1}_{c_2 b}(-1)^{\sum_{j=2}^{L}([\alpha_j]+[\beta_j])\sum_{i=1}^{j-1}\epsilon_{\alpha_i}}.$$

We call $T(x)$ the monodromy matrix and by construction it fulfills the same intertwining relation as the L operators.

The transfer matrix of the integrable model is given as the supertrace of the monodromy matrix. This operator is given by

$$\tau(x) = \mathrm{str}[T(x)] = \sum_i (-1)^{[i]} T(x)_{ii}.$$

The $\tau(x)$ form a one-parameter family of commuting operators. The transfer matrix may be taken as integrals of the motion and we can obtain an infinite number of conservation laws of the model. It can be employed to construct exactly solvable models in the usual way.

4. Algebraic Bethe ansatz with BBF grading

We use the matrix from the vector representation as our R-matrix and the L operator given above for obtaining the defining equations for the algebra constructed from (2). Represent the monodromy matrix in the following way

$$\begin{aligned}
T_L(x) &= L_L(x)L_{L-1}(x)\ldots L_1(x) \\
&= \begin{bmatrix} T_{11}(x) & T_{12}(x) & T_{13}(x) \\ T_{21}(x) & T_{22}(x) & T_{23}(x) \\ T_{31}(x) & T_{32}(x) & T_{33}(x) \end{bmatrix}.
\end{aligned} \tag{7}$$

The transfer matrix is given by

$$\tau(y) = \mathrm{str}[T_L(y)] = T_{11}(y) + T_{22}(y) - T_{33}(y).$$

Take the lowest weight state as a reference state (pseudo-vacuum) in W, which we denote as $|0\rangle_k$; that is, $|0\rangle_k = |d\rangle$. Then the action of $L_k(x)$ on the reference state on the kth site is

$$L(x)|0\rangle_k = \begin{pmatrix} I(x) & 0 & 0 \\ 0 & I(x) & 0 \\ * & ** & 1 \end{pmatrix}|0\rangle_k.$$

$*$ and $**$ represent complicated values that are not necessary to evaluate and

$$I(x) = \frac{(xq^{-\alpha} - 1)}{(xq^{\alpha+1} - q^{-1})}.$$

We find the action of the monodromy matrix on the reference state to be given by

$$T_L(x)|0\rangle = \begin{pmatrix} I(x)^L & 0 & 0 \\ 0 & I(x)^L & 0 \\ T_{31}(x) & T_{32}(x) & 1 \end{pmatrix} |0\rangle. \tag{8}$$

We construct a set of eigenstates of the transfer matrix using the technique of the NABA. The creation operators are $T_{31}(x)$, $T_{32}(x)$ due to the choice of reference state. Thus we use the following for the ansatz for the eigenstates of $\tau(y)$:

$$|x_1, ..., x_n|F\rangle = T_{3a_1}(x_1)T_{3a_2}(x_2)...T_{3a_n}(x_n)|0\rangle F^{a_n...a_1} \tag{9}$$

where indices a_i have values 1 or 2 and $F^{a_n...a_1}$ is a function of the spectral parameters x_j. The action of these states is determined by the monodromy matrix and the relations (2) which in essence determine the quantum Yangian $Y_q[gl(2|1)]$ [15]. The relations necessary for the construction of the NABA are

$$T_{33}(y)T_{3a}(x) = -\frac{1}{E(x/y)}T_{3a}(x)T_{33}(y) + x/y\frac{C(x/y)}{E(x/y)}T_{3a}(y)T_{33}(x) \tag{10}$$

$$T_{ab}(y)T_{3c}(x) = \frac{r_{cp,bd}(y/x)}{E(y/x)}T_{3p}(x)T_{ad}(y) + \frac{C(y/x)}{E(y/x)}T_{3b}(y)T_{ac}(x) \tag{11}$$

$$T_{3a_1}(x_1)T_{3a_2}(x_2) = r_{a_2b_2,a_1b_1}(x_1/x_2)T_{3b_2}(x_2)T_{3b_1}(x_1) \tag{12}$$

where

$$r(y) = \begin{bmatrix} A(y) & 0 & 0 & 0 \\ 0 & E(y) & C(y) & 0 \\ 0 & yC(y) & E(y) & 0 \\ 0 & 0 & 0 & A(y) \end{bmatrix}.$$

Since $[1] = [2] = 0$, this R-matrix is essentially not graded and it can be seen that $r(y)$ also fulfills a Yang–Baxter equation and can be identified with the R-matrix of the quantum spin $\frac{1}{2}$ Heisenberg (XXZ) model. The diagonal elements of the monodromy matrix act on the states in the following way:

$$\begin{aligned} T_{33}(y)|x_1, ..., x_n|F\rangle &= (-1)^n \prod_{i=1}^{n} \frac{1}{E\,(x_i/y)}|x_1, ..., x_n|F\rangle \\ &\quad + \sum_{k=1}^{n} (\breve{\Lambda}_k)^{b_1...b_n}_{a_1...a_n} T_{3b_k}(y) \prod_{j=1, j\neq k}^{n} T_{3b_j}(x_j)|0\rangle F^{a_n...a_1} \end{aligned} \tag{13}$$

$$\begin{aligned} [T_{11}(y) + T_{22}(y)]|x_1, ..., x_n|F\rangle &= I(y)^L \prod_{j=1}^{n} \frac{1}{E\left(y/x_j\right)} \prod_{l=1}^{n} T_{3b_l}(x_l)|0\rangle \tau^{(1)}(y)^{b_1...b_n}_{a_1...a_n} F^{a_n...a_1} \\ &\quad + \sum_{k=1}^{n} (\Lambda_k)^{b_1...b_n}_{a_1...a_n} T_{3b_k}(y) \prod_{j=1, j\neq k}^{n} T_{3b_j}(x_j)|0\rangle F^{a_n...a_1} \end{aligned} \tag{14}$$

where $I(y)$ is defined above and

$$\tau^{(1)}(y)^{b_1...b_n}_{a_1...a_n} = \mathrm{str}[T^{(1)}_n(y)] \tag{15}$$

that is

$$\tau^{(1)}(y)^{b_1\ldots b_n}_{a_1\ldots a_n} = str\left[L^{(1)}_n\left(y/x_n\right)L^{(1)}_{n-1}\left(y/x_{n-1}\right)\ldots L^{(1)}_2\left(y/x_2\right)L^{(1)}_1\left(y/x_1\right)\right].$$

We have

$$[L^{(1)}_k(x)]_{ij} = \sum r_{ik,jl}e_{kl}. \tag{16}$$

So the operators $L^{(1)}$ and $r(y)$ can be interpreted as the L operator and R-matrix of the XXZ model. Hence $T^{(1)}_n(y)$ and $\tau^{(1)}(y)$ are the monodromy and transfer matrices for the corresponding model with inhomogeneities x_i, $i = 1, \ldots, n$. The eigenvalue condition

$$\tau(y)|x_1, \ldots, x_n|F\rangle = \mu(y, \{x_j\}, F)|x_1, \ldots, x_n|F\rangle$$

leads to the requirement that F be an eigenvector of the nested transfer matrix $\tau^{(1)}(y)$, and that the unwanted terms, Λ_k, $\breve{\Lambda}_k$ cancel. That is,

$$[(\Lambda_k)^{b_1\ldots b_n}_{a_1\ldots a_n} - (\breve{\Lambda}_k)^{b_1\ldots b_n}_{a_1\ldots a_n}]F^{a_n\ldots a_1} = 0.$$

These values are computed in appendix A. This leads us to the conditions on the spectral parameters x_j and coefficients F, necessary for eigenvalue condition to hold. This equation simplifies to

$$[I(x_k)]^{-L}(-1)^n\prod_{i=1,i\neq k}^{n}\frac{E(x_k/x_i)}{E(x_i/x_k)}F^{b_n\ldots b_1} = \tau^{(1)}(x_k)^{b_1\ldots b_n}_{a_1\ldots a_n}F^{a_n\ldots a_1}$$
$$k = 1, \ldots, n. \tag{17}$$

The next step in the NABA is to solve the nesting. The condition that F be an eigenvector of $\tau^{(1)}(y)$ requires the diagonalization of $\tau^{(1)}(y)$, which can be acheived by performing a second, nested Bethe ansatz. We write the monodromy and transfer matrices as follows,

$$T^{(1)}_n(y) = \begin{bmatrix} T^{(1)}_{11}(y) & T^{(1)}_{12}(y) \\ T^{(1)}_{21}(y) & T^{(1)}_{22}(y) \end{bmatrix} \tag{18}$$

$$\tau^{(1)}(y) = T^{(1)}_{11}(y) + T^{(1)}_{22}(y).$$

Obtaining, as before, the equations from the relation (2) necessary for the NABA we have

$$T^{(1)}_{11}(y)T^{(1)}_{21}(x) = \frac{A(y/x)}{E(y/x)}T^{(1)}_{21}(x)T^{(1)}_{11}(y) - \frac{C(y/x)}{E(y/x)}T^{(1)}_{21}(y)T^{(1)}_{11}(x) \tag{19}$$

$$T^{(1)}_{22}(y)T^{(1)}_{21}(x) = \frac{A(x/y)}{E(x/y)}T^{(1)}_{21}(x)T^{(1)}_{22}(y) - x/y\frac{C(x/y)}{E(x/y)}T^{(1)}_{21}(y)T^{(1)}_{22}(x) \tag{20}$$

$$T^{(1)}_{21}(x)T^{(1)}_{21}(y) = T^{(1)}_{21}(y)T^{(1)}_{21}(x). \tag{21}$$

For the reference states, choose

$$|0\rangle^{(1)}_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |0\rangle^{(1)} = \otimes^n_{k=1}|0\rangle^{(1)}_k.$$

The action of the nested monodromy matrix $T^{(1)}(y)$ on the reference state is

$$T^{(1)}_{11}(y)|0\rangle^{(1)} = \prod_{j=1}^{n}E\left(y/x_j\right)|0\rangle^{(1)}$$

$$T^{(1)}_{22}(y)|0\rangle^{(1)} = \prod_{j=1}^{n}A\left(y/x_j\right)|0\rangle^{(1)}. \tag{22}$$

We choose the following ansatz for the eigenstates of $\tau^{(1)}(y)$:

$$|x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle = T_{21}^{(1)}(x_1^{(1)}), \ldots, T_{21}^{(1)}(x_{n_1}^{(1)})|0\rangle^{(1)}.$$

These states can be related to the coefficients $F^{a_n \ldots a_1}$ by noting that the state $|x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle$ exists on a lattice of n sites and is thus an element of a direct product over n two-dimensional Hilbert spaces. The action of $\tau^{(1)}(y)$ on the states is computed as before from the relations (2). We obtain

$$T_{22}^{(1)}(y)|x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle = \prod_{i=1}^{n_1} \frac{A\left(x_i^{(1)}/y\right)}{E\left(x_i^{(1)}/y\right)} \prod_{j=1}^{n} A\left(y/x_j\right)|x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle + \sum_{k=1}^{n_1} (\breve{\Lambda}_k)^{(1)} T_{21}^{(1)}(y) \prod_{j=1, j\neq k}^{n_1} T_{21}^{(1)}(x_j)|0\rangle^{(1)} \tag{23}$$

$$T_{11}^{(1)}(y)|x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle = \prod_{i=1}^{n_1} \frac{A\left(y/x_i^{(1)}\right)}{E\left(y/x_i^{(1)}\right)} \prod_{j=1}^{n} E\left(y/x_j\right)|x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle + \sum_{k=1}^{n_1} (\Lambda_k)^{(1)} T_{21}^{(1)}(y) \prod_{j=1, j\neq k}^{n_1} T_{21}^{(1)}(x_j)|0\rangle^{(1)}. \tag{24}$$

The eigenvalues for $\tau^{(1)}(y)$ are found to be

$$\tau^{(1)}(y)|x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle = \left[\prod_{i=1}^{n_1} \frac{A\left(y/x_i^{(1)}\right)}{E\left(y/x_i^{(1)}\right)} \prod_{j=1}^{n} E\left(y/x_j\right) + \prod_{i=1}^{n_1} \frac{A\left(x_i^{(1)}/y\right)}{E\left(x_i^{(1)}/y\right)} \prod_{j=1}^{n} A\left(y/x_j\right)\right] \times |x_1^{(1)}, \ldots, x_{n_1}^{(1)}\rangle. \tag{25}$$

Inserting this into equation (17) for $y = x_k$ we have the first of the Bethe equations

$$(-1)^n I(x_k)^{-L} \prod_{j=1, j\neq k}^{n} \frac{E\left(x_k/x_j\right)}{E\left(x_j/x_k\right)} = \prod_{j=1}^{n_1} \frac{A\left(x_j^{(1)}/x_k\right)}{E\left(x_j^{(1)}/x_k\right)} \prod_{i=1, i\neq k}^{n} A\left(x_k/x_i\right)$$
$$k = 1, \ldots, n. \tag{26}$$

To ensure that we have the eigenstates of the transfer matrix for the nesting we must have the unwanted terms $\breve{\Lambda}_k^{(1)}, \Lambda_k^{(1)}$ cancelling and these values are computed in the appendix. The resulting equations after some simplification is the set of Bethe equations for the nesting as follows:

$$\prod_{j=1, j\neq k}^{n_1} \frac{A\left(x_k^{(1)}/x_j^{(1)}\right)}{E\left(x_k^{(1)}/x_j^{(1)}\right)} \prod_{i=1}^{n} E\left(x_k^{(1)}/x_i\right) = \prod_{i=1}^{n_1} \frac{A\left(x_i^{(1)}/x_k^{(1)}\right)}{E\left(x_i^{(1)}/x_k^{(1)}\right)} \prod_{j=1}^{n} A\left(x_k^{(1)}/x_j\right)$$
$$k = 1, \ldots, n_1. \tag{27}$$

n and n_1 can be identified as the total number of electrons (N_e) and the number of spin-down electrons ($N_\downarrow$) respectively. With some substitution and simplification the Bethe equations reduce to the following

$$\left[\frac{(xq^{-\alpha}-1)}{(xq^{\alpha+1}-q^{-1})}\right]^{-L} = \prod_{j=1,j\neq k}^{N_\downarrow} \frac{x_k - x_j^{(1)}q^2}{q(x_k - x_j^{(1)})}$$
$$k = 1, \ldots, N_e \tag{28}$$

$$\prod_{j=1,j\neq k}^{N_\downarrow} \frac{-x_i^{(1)} + x_k^{(1)}q^2}{x_k^{(1)} - x_j^{(1)}q^2} = \prod_{i=1}^{N_e} \frac{x_i - x_k^{(1)}q^2}{q(x_i - x_k^{(1)})} \qquad k = 1, \ldots, N_\downarrow. \tag{29}$$

The eigenvalues of the transfer matrix are given by

$$\mu(y, \{x_j\}, F) = \left[\frac{(xq^{-\alpha}-1)}{(xq^{\alpha+1}-q^{-1})}\right]^{L} \prod_{j=1}^{N_e} \frac{y - x_j q^2}{q(x_j - y)} \mu^{(1)}(y) - \prod_{i=1}^{N_e} \frac{x_i - yq^2}{q(x_i - y)} \tag{30}$$

$$\mu^{(1)}(y) = \prod_{i=1}^{N_\downarrow} \frac{x_i^{(1)} - yq^2}{q(x_i^{(1)} - y)} \prod_{j=1}^{N_e} \frac{(x_j - y)q}{y - x_j q^2} + \prod_{i=1}^{N_\downarrow} \frac{y - x_i^{(1)}q^2}{q(y - x_i^{(1)})} \prod_{j=1}^{N_e} \frac{x_j - yq^2}{y - x_j q^2}. \tag{31}$$

5. Conclusion

This model has been solved previously in [13] with the use of the co-ordinate Bethe ansatz. The solution presented in [13] (see equations 6) is the same as the solution obtained above (28, 29) with $x = e^{ic}$, $q = e^{id}$ and the following substitutions:

$$\lambda_j = \frac{1}{2}(c_k + d) \qquad a = -\frac{i}{2}d(\alpha + 1)$$

$$\Lambda_\alpha = \frac{1}{2}(c_j^{(1)} + 2d) \qquad \gamma = id.$$

Similarly the equations (71, 72) in [2] may be obtained with the substitutions:

$$u_k = i(c_k + d) \qquad v_j = i(c_j^{(1)} + 2d)$$

$$\gamma = -d \qquad b = -(\alpha + \frac{1}{2}).$$

Substituting $x = q^\theta$ and taking the limit $q \to 1$ in the BAE, we can easily obtain the BAE for the $gl(2|1)$ model in [9]. The quantum version of this solution on the open chain will be presented in a future publication.

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Appendix

Here we calculate the unwanted terms following the method set out in [6]. The unwanted terms are identified by containing a creation operator with spectral parameter y. The cancellation of the unwanted terms ensures that the states (9) are eigenstates of the transfer matrix $\tau(y)$. To determine $\breve{\Lambda}_k$, it is convenient to commute the first creation operator with spectral parameter λ_k to the first place in the ansatz using the commutation rule extracted from the relations arising from (2). That is, we write

$$\prod_{i=1}^{n} T_{3a_i}(x_i) = T_{3b_k}(x_k) \prod_{i=1}^{k-1} T_{3b_i}(x_i) \prod_{j=k+1}^{n} T_{3a_j}(x_j) S(x_k)^{b_1\ldots b_k}_{a_1\ldots a_k}$$

$$S(x_k)^{b_1\ldots b_k}_{a_1\ldots a_k} = r\,(x_{k-1}/x_k)^{a_k c_{k-1}}_{a_{k-1} b_{k-1}}\, r\,(x_{k-2}/x_k)^{c_{k-1} c_{k-2}}_{a_{k-2} b_{k-2}} \ldots r\,(x_1/x_k)^{c_2 b_k}_{a_1 b_1}\,. \tag{A.1}$$

To obtain an unwanted term, we commute T_{33} past $T_{3b_k}(x_k)$ using the second term in (10) then we use the first term to commute $T_{33}(x_k)$ with the other terms in the ansatz until it acts on the vacuum according to (8). The resulting equation for $\breve{\Lambda}_k$ is

$$\breve{\Lambda}_k F^{b_1\ldots b_n} = S(x_k)^{b_1\ldots b_k}_{a_1\ldots a_k} F^{b_n\ldots b_{k+1} a_k\ldots a_1} \left[\frac{x_k C\,(x_k/y)}{y E\,(x_k/y)}\right] (-1)^{n-1} \prod_{j=1, j\neq k}^{n} \frac{1}{E\left(x_j/x_k\right)}.$$

For Λ_k, we have two terms involved. So we write $\Lambda_k = \Lambda_{k,1} + \Lambda_{k,2}$; these terms arise from $T_{11}(y)$ and $T_{22}(y)$ respectively. With similar working to that for $\breve{\Lambda}_k$ we find the contribution from the $T_{11}(y)$ terms to be

$$\Lambda_{k,1} F^{b_1\ldots b_n} = S(x_k)^{c_1\ldots c_k}_{a_1\ldots a_k} F^{a_n\ldots a_1} \left[\frac{C(y/x_k)}{E(y/x_k)}\right] \delta_{b_k,1}\delta_{d_{n-1},1} I(x_k)^L \prod_{j=1, j\neq k}^{n} \frac{1}{E(x_k/x_j)}$$
$$\times r\,(x_k/x_1)^{c_1 b_1}_{c_k d_1}\, r\,(x_k/x_2)^{c_2 b_2}_{d_1 d_2} \ldots r\,(x_k/x_{k-1})^{c_{k-1} b_{k-1}}_{d_{k-2} d_{k-1}}$$
$$\times r\,(x_k/x_{k+1})^{a_{k+1} b_{k+1}}_{d_{k-1} d_k}\, r\,(x_k/x_{k+2})^{a_{k+2} b_{k-1}}_{d_k d_{k+1}} \ldots r\,(x_k/x_n)^{a_n b_n}_{d_{n-1} d_{n-2}}\,. \tag{A.2}$$

The delta functions appearing in this equation arise in the following way. Commuting $T_{11}(x_k)$ past the terms of the ansatz to the vacuum leads us to the term $\delta_{d_{n-1},1}$. $\delta_{b_k,1}$ is necessary as in (14) we need identify the $T_{3b_k}(y)$ term with the $T_{11}(y)$ contributions. The contribution from the $T_{22}(y)$ are obtained similarly with the factors $\delta_{b_k,2}, \delta_{d_{n-1},2}$ being the only difference between $\Lambda_{k,1}$ and $\Lambda_{k,2}$. Then with $\Lambda_k = \Lambda_{k,1} + \Lambda_{k,2}$ we have

$$\Lambda_k F^{b_1\ldots b_n} = S(x_k)^{c_1\ldots c_k}_{a_1\ldots a_k} F^{a_n\ldots a_1} \left[\frac{C(y/x_k)}{E(y/x_k)}\right] I(x_k)^L \prod_{j=1, j\neq k}^{n} \frac{1}{E(x_k/x_j)}$$
$$\times r\,(x_k/x_1)^{c_1 b_1}_{c_k d_1}\, r\,(x_k/x_2)^{c_2 b_2}_{d_1 d_2} \ldots r\,(x_k/x_{k-1})^{c_{k-1} b_{k-1}}_{d_{k-2} d_{k-1}}$$
$$\times r\,(x_k/x_{k+1})^{a_{k+1} b_{k+1}}_{d_{k-1} d_k}\, r\,(x_k/x_{k+2})^{a_{k+2} b_{k-1}}_{d_k d_{k+1}} \ldots r\,(x_k/x_n)^{a_n b_n}_{d_{n-2} b_k}\,. \tag{A.3}$$

We may simplify this equation by contracting the $c_1 \ldots c_n$ indices using the unitarity of the r-matrix . That is,

$$r\,(x_1/x_2)\, r^T\,(x_2/x_1) = I$$

or in component form

$$r\,(x_1/x_2)^{b_1 a_1}_{b_2 a_2}\, r\,(x_2/x_1)^{c_1 b_1}_{c_2 b_2} = \delta_{a_1 c_1}\delta_{a_2 c_2}.$$

As a result the following terms in (A.2) may be simplified

$$S(x_k)^{c_1\ldots c_k}_{a_1\ldots a_k} r\,(x_k/x_1)^{b_1 c_1}_{d_1 c_k} \ldots r\,(x_k/x_{k-1})^{b_{k-1} c_{k-1}}_{d_{k-1} d_{k-2}} = \prod_{i=1}^{k-1} \delta_{a_i, d_i}\delta_{b_{k-1}, a_k}.$$

The remaining r-matrices we convert into L operators according to

$$[L^{(1)}(x)]_{ij} = \sum r(x)_{ik,jl} e_{kl}.$$

The unwanted terms are written as

$$\Lambda_k F^{b_1 \dots b_n} = \left[\frac{C(y/x_k)}{E(y/x_k)}\right] I(x_k)^L \prod_{i=1, i\neq k}^{n} \frac{1}{E(x_k/x_i)} F^{a_n \dots a_k b_{k-1} \dots b_1}$$
$$\times L_n^{(1)}(x_k/x_n)_{d_{n-2} b_k}^{a_n b_n} L_{n-1}^{(1)}(x_k/x_{n-1})_{d_{n-3} d_{n-2}}^{a_{n-1} b_{n-1}} \dots L_{k+1}^{(1)}(x_k/x_{k+1})_{a_k d_k}^{a_{k+1} b_{k+1}}. \tag{A.4}$$

We now insert the equations for $\breve{\Lambda}_k$ and Λ_k into the equation for the cancellation of unwanted terms and multiply throughout by $S^{(-1)}(x_k)$. We note once again using the unitarity of $r(y)$ that we have

$$S^{(-1)}(x_k)_{b_1 \dots b_k}^{p_1 \dots p_k} S(x_k)_{a_1 \dots a_k}^{b_1 \dots b_k} = \prod_{i=1}^{k} \delta_{a_i, p_i}.$$

The result after some simplification is

$$I(x_k)^{(-L)}(-1)^n \prod_{i=1, i\neq k}^{n} \frac{E(x_k/x_i)}{E(x_i/x_k)} F^{b_n \dots b_{k+1} p_k \dots p_1} = [\tau^{(1)}(x_k) F]^{b_n \dots b_{k+1} p_k \dots p_1}.$$

Working in a similar manner to the above for the computation of the unwanted terms for the nested case leads to the equations

$$\breve{\Lambda}_k^{(1)} = -\frac{x_k^{(1)} C\left(x_k^{(1)}/y\right)}{y E\left(x_k^{(1)}/y\right)} \prod_{j=1, j\neq k}^{n_1} \frac{A\left(x_j^{(1)}/x_k^{(1)}\right)}{E\left(x_j^{(1)}/x_k^{(1)}\right)} \prod_{i=1}^{n} A\left(x_k^{(1)}/x_i\right) \tag{A.5}$$

$$\Lambda_k^{(1)} = -\frac{C\left(y/x_k^{(1)}\right)}{E\left(y/x_k^{(1)}\right)} \prod_{j=1, j\neq k}^{n_1} \frac{A\left(x_k^{(1)}/x_j^{(1)}\right)}{E\left(x_k^{(1)}/x_j^{(1)}\right)} \prod_{i=1}^{n} E\left(x_k^{(1)}/x_i\right). \tag{A.6}$$

The cancellation of $\Lambda_k^{(1)}$ and $\breve{\Lambda}_k^{(1)}$ leads to the equation

$$\prod_{j=1, j\neq s}^{n_1} \frac{A\left(x_s^{(1)}/x_j^{(1)}\right)}{E\left(x_s^{(1)}/x_j^{(1)}\right)} \prod_{i=1}^{n} E\left(x_s^{(1)}/x_i\right) = \prod_{i=1}^{n_1} \frac{A\left(x_i^{(1)}/x_s^{(1)}\right)}{E\left(x_i^{(1)}/x_s^{(1)}\right)} \prod_{p=1}^{n} A\left(x_s^{(1)}/x_p\right)$$
$$s = 1, \dots, n_1. \tag{A.7}$$

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